Overruled Lagrangian Cones: Origins and Incarnations

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Abstract

Overruled Lagrangian cones arise in practice as the graphs of the differentials of generating functions of genus 0 Gromov-Witten invariants. This paper provides a detailed account of the proof of this fact given in [2], along with all background knowledge needed to understand it. We then turn to describing the set of overruled Lagrangian cones. According to [1], an overruled Lagrangian cone can be reconstructed from a single point, but it is hard to tell if reconstruction on a given point will yield an overruled Lagrangian cone. This paper takes the route of starting with a single ruling space. Ruling spaces are $\mathcal{D}$-modules, and one can be specified by the connection describing its $\mathcal{D}$-module structure. Reconstruction turns into an action of a lie algebra, consisting of differential operators in Novikov's variables, on the space of connections. Overruled Lagrangian cones are orbits of this action.

Introduction

Let H be a finite dimensional vector space over k with a non-degenerate symmetric inner product and let $\mathcal{H} = H((z))$. Consider the decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ given by decomposing a Laurent series into its power series and polar parts respectively. Let π_+ and π_- be the projections onto $\mathcal{H}_+$ and $\mathcal{H}_-$. Endow $\mathcal{H}$ with the symplectic form

$$\Omega(f,g) = Res_{z=0}(f(z), g(-z))$$

where $f, g \in \mathcal{H}$ and $(,)$ denotes the inner product on H. $(\mathcal{H}, \Omega)$ is the symplectic loop space of H.

Definition. *An overruled Lagrangian cone $\mathcal{L} \in \mathcal{H}$ is a Lagrangian submanifold whose tangent spaces are $k[[z]]$ submodules, that project isomorphically to $\mathcal{H}_+$ under π_+, such that each tangent space T is tangent to $\mathcal{L}$ along zT.*

The first part of this paper, origins, describes in detail how overruled Lagrangian cones arise from Gromov Witten theory, which is intersection theory on moduli spaces of holomorphic curves in a given target space. A certain class of intersection numbers on moduli space, for a given genus g, are encoded in a generating function called the genus g descendant potential. The descendant potential is a formal function on $\mathcal{H}_+$, and the graph of the differential of the genus 0 descendant potential turns out to be an overruled Lagrangian cone in $\mathcal{H}$. A substantial section of this paper is devoted to proving this fact. This encoding of descendants is well known in the literature, and has proved very useful. This paper provides a self contained and detailed exposition that assumes little prior knowledge. I chose to follow the proof in the appendix of [2] because it is conceptually very interesting. One defines a family of modified generating functions, parameterized by $H^*(X)$, called ancestor potentials, then the relationship between ancestor potentials and the descendant potential proves overruledness. Interestingly, the formalism of quantization plays a key role in the proof. The genus 0 theory is somehow the classical limit of the theory of all genera.

The second part of this paper, incarnations, is original joint work of Alexander Givental and I, culminating in an alternative description of what an overruled Lagrangian cone is. Before summarizing this, it should be mentioned that we will be dealing with overruled Lagrangian cones, not over a field, but over a power series ring $R = \mathbb{Q}[[Q_1, ..., Q_r]]$. We can either view R as the ground ring, or imagine a family of overruled Lagrangian cones over $\operatorname{spec}(R)$. Whenever we talk about Laurent series or polynomials over R, they are actually only required to polynomial or Laurent modulo all powers of the maximal ideal of R. The overruled Lagrangian cones of interest have the additional property that their tangent spaces are $\mathcal{D}$-modules, where $\mathcal{D}$ is, roughly speaking, an algebra of differential operators in $Q_1, ..., Q_r$. Each tangent space can be specified by a connection, which in our case, concretely is a tuple of r matrices with coefficients in R satisfying some properties. An overruled Lagrangian cone will correspond to a family of connections, that is an integral of a very interesting distribution.

Origins

Gromov Witten Invariants

Let X be a compact Kähler manifold and let $X_{g,n,d}$ be the Kontsevich moduli space of degree d stable maps of genus g curves with n marked points to X [8]. Here d is a class in $H_2(X, \mathbb{Z})$ which is representable by a holomorphic curve. For each marked point there is an evaluation map $ev_i : X_{g,n,d} \to X$ given by evaluation at the i'th marked point. The pullbacks of cohomology classes on X by these evaluation maps are important cohomology classes on $X_{g,n,d}$ which we will consider in our intersection theory. Classical enumerative geometry questions of the form "how many genus g, degree d curves pass through the cycles $Y_1, ..., Y_l \subset X$?" can be answered by intersecting these classes.

The other cohomology classes we consider are called ψ_i. Let $\pi : X_{g,n+1,d} \to X_{g,n,d}$ be the map that forgets the $n+1'st$ marked point. With this projection we may interpret $X_{g,n+1,d}$ as the universal curve over $X_{g,n,d}$. Sometimes $X_{g,n+1,d}$ is written $C_{g,n,d}$ to emphasize this interpretation. Let $s_1, ..., s_n : X_{g,n,d} \to C_{g,n,d}$ denote the sections corresponding to marked points 1 through n. Let L_i be the line bundle who's fiber at a curve is the cotangent space to that curve at the i'th marked point. Formally L_i is the conormal bundle of s_i. Let ψ_i be the first Chern class of L_i. Intersection numbers of the classes ψ_i and $ev_i^*(\phi)$ are called gravitational descendants (according to Witten, they are related to two dimensional gravity [9]) and they will be our main object of study.

Universal Relations

Gravitational Descendants satisfy a few interesting relations which we now derive. The same symbols will be used to denote L_i and ψ_i on each $X_{g,n,d}$, and π will always denote forgetting the last marked point, so the meaning of these symbols depends on context. We will usually understand cohomology classes via Poincare dual cycles. The following results are literally true if X is convex, that is each tangent space is spanned by global vector fields. In the non-convex case, one must construct *virtual fundamental classes* [7] for $X_{g,n,d}$ for our results to hold. Everything rests on the following lemma which compares $\psi_i \in X_{g,n,d}$ and $\psi_i \in X_{g,n+1,d}$. Let $[s_i]$ denote the cohomology class Poincare dual to s_i.

Lemma (Comparison Lemma)**.**

$$\psi_i = \pi^* \psi_i + [s_i]$$

Proof. We have a map $\pi^* L_i \to L_i$ given by the differential of the map of universal curves $C_{g,n+1,d} \to C_{g,n,d}$. This differential is an isomorphism, except when the $i'th$ point is on a component that collapses, in which case it is zero. This happens when the $i'th$ and $n+1'st$ marked points lie on an irreducible component that is mapped to a point in X. Such curves are precisely those over s_i, thus

$$\psi_i = \pi^* \psi_i + n[s_i]$$

for some integer n. Note that $s_i^* L_i$ is trivial whereas $s_i^* \pi^* L_i = L_i$, so

$$s_i^*(L_i \otimes (\pi^* L_i)^{-1}) = L_i^{-1}$$

L_i^{-1} is the normal bundle of s_i so its Chern class is $s_i^*[s_i]$. Taking the Chern classes we get.

$$s_i^* \psi_i - s_i^* \pi^* \psi_i = s_i^*[s_i]$$

Which implies that $n = 1$. □

Note that forgetting points and evaluation commute, so $\pi^* ev^*(u) = ev^*(u)$.

Theorem 1 (String Equation)**.** *If $X_{g,n,d}$ is non-empty than*

$$\int_{X_{g,n+1,d}} \psi_1^{k_1} \cdots \psi_n^{k_n} ev_1^*(\phi_1) \cdots ev_n^*(\phi_n) = \sum_{i=1}^{n} \int_{X_{g,n,d}} \psi_1^{k_1} \cdots \psi_i^{k_i - 1} \cdots \psi_n^{k_n} ev_1^*(u_1) \cdots ev_n^*(u_n)$$

Proof. ψ_i and $[s_i]$ have intersection product zero because $s_i^* L_i$ is trivial so we have

$$(\pi^* \psi_i)^n = (\psi_i^n - D_{i,n+1})^n = \psi_i^n + [-D_{i,n+1}]^n$$

The pullback of a class on $X_{g,n,d}$ to $X_{g,n+1,d}$ will have dimension greater than zero so we have

$$0 = \int_{X_{g,n+1,d}} \pi^* \psi_1^{k_1} \cdots \pi^* \psi_n^{k_n} ev_1^*(u_1) \cdots ev_n^*(u_n)$$

$$= \int_{X_{g,n+1,d}} (\psi_1^{k_1} + (-[s_1])^{k_1}) \cdots (\psi_n^{k_n} + (-[s_n])^{k_n}) ev_1^*(u_1) \cdots ev_n^*(u_n)$$

The hyperplanes s_i do not intersect one another so non vanishing terms can have at most one.

$$0 = \int_{X_{g,n+1,d}} \psi_1^{k_1} \cdots \psi_n^{k_n} ev_1^*(u_1) \cdots ev_n^*(u_n) + \sum_{i=1}^{n} \int_{X_{g,n+1,d}} \psi_1^{k_1} \cdots (-[s_i])^{k_i} \cdots \psi_n^{k_n} ev_1^*(u_1) \cdots ev_n^*(u_n)$$

The result follows from interpreting one copy of $[s_i]$ in the integrand as pulling the rest of the integrand back to $X_{g,n,d}$ by s_i. Recall that $s_i^*[-s_i] = \psi_i$. $\square$

Theorem 2 (Dilaton Equation). *The pushforward of ψ_{n+1} by π is $2g-2+n$. This leads to*

$$\int_{X_{g,n+1,d}} \psi_1^{k_1} \cdots \psi_n^{k_n} \psi_{n+1} ev_1^*(u_1) \cdots ev_n^*(u_n) = (2g-2+n) \int_{X_{g,n,d}} \psi_1^{k_1} \cdots \psi_n^{k_n} ev_1^*(u_1) \cdots ev_n^*(u_n)$$

Proof. It is equivalent but more convenient to compute the pushforward of ψ_1 by π_1 where π_1 is the map that forgets the first marked point. $(\pi^*)^n L_1$ on $X_{g,n+1,d}$ is isomorphic to the relative cotangent bundle of $\pi_1 : X_{g,n+1,d} \to X_{g,n,d}$. Smooth fibers of π_1 are genus g curves and a section of the relative cotangent bundle defines a 1-form on each fiber so will vanish $2g-2$ times on each smooth fiber. This means $(\pi^*)^n \psi_1$ has intersection number $2g-2$ with each smooth fiber. Iterative application of the comparison lemma gives

$$\psi_1 = (\pi^*)^n \psi_1 + (\pi^*)^{n-1}[s_1] + (\pi^*)^{n-2}[s_1] + \cdots + [s_1]$$

A smooth fiber of π_1 will thus have intersection number $2g-2+n$ with ψ_1 so $\pi_* \psi_{n+1} = (2g-2+n)$. This means

$$\int_{X_{g,n+1,d}} \pi^*(\psi_1^{k_1} \cdots \psi_n^{k_n}) \psi_{n+1} ev_1^*(u_1) \cdots ev_n^*(u_n) = (2g-2+n) \int_{X_{g,n,d}} \psi_1^{k_1} \cdots \psi_n^{k_n} ev_1^*(u_1) \cdots ev_n^*(u_n)$$

Since $s_i^* L_n + 1$ is trivial, the cup products $[s_i]\psi_{n+1}$ are zero so using the comparison lemma gives the result. $\square$

Theorem 3 (Divisor Equation). *The pushforward of $ev_{n+1}^*(u)$ by π is (u,d). This leads to*

$$\int_{X_{g,n+1,d}} \psi_1^{k_1} \ldots \psi_n^{k_n} ev_1^*(u_1) \ldots ev_n^*(u_n) ev_{n+1}^*(u) = (u,d) \int_{X_{g,n,d}} \psi_1^{k_1} \ldots \psi_n^{k_n} ev_1^*(u_1) \ldots ev_n^*(u_n)$$

$$+ \sum_{i=1}^{n} \int_{X_{g,n,d}} \psi_1^{k_1} \cdots \psi_i^{k_i - 1} \cdots \psi_n^{k_n} ev_1^*(u_1) \cdots ev_i^*(uu_i) \cdots ev_n^*(u_n)$$

Note that this is a generalization of the string equation.

Proof. Let $\Sigma \in X_{g,n,d}$. Its image will have intersection number (u,d) with u so there are (u,d) places where we can add a marked point such that it will land on u. This means that there are (u,d) points in $\pi^{-1}(\Sigma)$ that lie on $ev_{n+1}^*(u)$ so the pushforward of $ev_{n+1}^*(u)$ is (u,d). On the image of s_i, the i'th and $n+1'st$ points get stuck together so $s_i \circ ev_{n+1} = ev_i$. It follows that $s_i^* ev_{n+1}^*(u) = ev_i^*(u)$. Using a similar argument as in the proof of the string equation shows the result.

Theorem 4 (WDVV Equation)**.** *Let $\{\phi_\alpha\}$ and $\{\phi^\alpha\}$ be dual bases of $H^*(X)$. The expression*

$$\sum_{d'+d''=d,\,\alpha} \int_{[X_{0,3,d'}]} ev_1^*(u_i)\psi_1^{k_i} ev_2^*(u_j)\psi_2^{k_i} ev_3^*(\phi_\alpha) \int_{[X_{0,3,d''}]} ev_1^*(u_k)\psi_1^{k_k} ev_2^*(u_l)\psi_2^{k_l} ev_3^*(\phi^\alpha)$$

is the same when i,j,k,l is any permutation of $1,2,3,4$.

Intuitively this is because we can glue two curves with 3 marked points together by their third marked points, then deform the curve so that the double point splits the four points into different pairings.

Proof. $M_{0,4}$ is isomorphic to CP_1 with three special points $0,1,\infty$ representing nodal curves with the three pairings of the four points. The isomorphism is given by the cross ratio. There is a map $ct : X_{0,4} \to M_{0,4}$ that sends a stable map to its underlying curve, contracting components that become unstable to points. Let Γ denote the pullback of the cohomology class of a point on $M_{0,4}$ by ct. We will show that the above integrand is equal to

$$\int_{X_{0,4,d}} \Gamma \prod_{i=1}^{4} ev_i^*(u_i)\psi_i^{k_i} = \int_{ct^{-1}(\gamma)} \prod_{i=1}^{4} ev_i^*(u_i)\psi_i^{k_i}$$

We will do the integral when $\gamma = 0,1,\infty$ so that it is over nodal curves. This nodal locus is isomorphic to the disjoint union, over all choices of d' and d'' which sum to d, of the preimages of the diagonal ΔX under the map

$$X_{0,3,d'} \times X_{0,3,d''} \xrightarrow{ev_3 \times ev_3} X \times X$$

The cohomology class Poincare dual to ΔX is $\sum_\alpha \phi_\alpha \otimes \phi^\alpha$. The result more or less follows. □

Degenerate Cases

There are two cases where $X_{g,n,d}$ exists but $X_{g,n-1,d}$ does not, namely $X_{0,3,0}$ and $X_{1,1,0}$. $X_{0,3,0}$ is isomorphic to X and all evaluation maps are identity. The line bundles L_1, L_2 and L_3 on $X_{0,3,0}$ are trivial so intersection numbers involving ψ_1,ψ_2, and ψ_3 are zero. If no ψ classes are included than we have

$$\int_{X_{0,3,0}} ev_1^*(u)ev_2^*(v)ev_3^*(w) = \int_X uvw$$

$X_{1,1,0}$ is isomorphic to $X \times M_{1,1}$ where $M_{1,1}$ is the moduli space of elliptic curves with one marked point. It turns out that $\int_{M_{1,1}} \psi_1 = \frac{1}{24}$ which has to do with the fact that $M_{1,1}$ is an orbifold [5]. We thus have that

$$\int_{X_{1,1,0}} \psi_1 ev_1^*(u)$$

is $\frac{1}{24}$ if u is a point and 0 otherwise.

□

Correlators

Correlator notation simplifies many formulas.

Definition.

$$\langle u_1\psi^{k_1},...,u_n\psi^{k_n}\rangle_{g,n,d} := \int_{[X_{g,n,d}]} ev_1^*(u_1)\psi_1^{k_1}\ldots ev_n^*(u_n)\psi_n^{k_n}$$

$$\langle u_1\psi^{k_1},...,u_n\psi^{k_n}\rangle_{g,n} := \sum_d Q^d\langle u_1\psi^{k_1},...,u_n\psi^{k_n}\rangle_{g,n,d}$$

Where $u_1,\ldots,u_n \in H^*(X)$.

The second type of correlator takes values in the *Novikov Ring* of X which is the power series completion of the semigroup ring of the cone $M \subset H_2(X,\mathbb{Z})$ of classes representable by holomorphic curves. It is generated by $\{Q^d : d \in M\}$ subject to $Q^dQ^{d'} = Q^{d+d'}$. Explicitly the Novikov ring is $\mathbb{C}[[Q^{d_1},...,Q^{d_r}]]$ where $d_1,...,d_r$ is a basis of M. We extend correlators to be multilinear functions over the Novikov ring, so they can take inputs of the form $\mathbf{t}(\psi)$where $\mathbf{t} \in \mathcal{H}_+$. In correlator notation our universal relations take the following form.

String Equation:
$\langle u_1\psi^{k_1},...,u_n\psi^{k_n},1\rangle_{g,n+1} = \sum_{i=1}^n\langle u_1\psi^{k_1},...,u_i\psi^{k_i-1},...,u_n\psi^{k_n}\rangle_{g,n}$

Dilaton Equation:
$\langle ...,\psi\rangle_{g,n+1} = (2g-2+n)\langle ...\rangle_{g,n}$

Divisor Equation:
$\langle u_1\psi^{k_1},...,u_n\psi^{k_n},u\rangle_{g,n+1,d} = (u,d)\langle u_1\psi^{k_1},...,u_n\psi^{k_n}\rangle_{g,n,d} + \sum_{i=1}^n\langle u_1\psi^{k_1},...,uu_i\psi^{k_i-1},...,u_n\psi^{k_n}\rangle_{g,n,d}$

WDVV Equation:
$\sum_\alpha\langle u_1\psi^{k_1},u_2\psi^{k_2},\phi_\alpha\rangle_{0,3}\langle\phi^\alpha,u_3\psi^{k_3},u_4\psi^{k_4}\rangle_{0,3} = \sum_\alpha\langle u_1\psi^{k_1},u_3\psi^{k_3},\phi_\alpha\rangle_{0,3}\langle\phi^\alpha,u_2\psi^{k_2},u_4\psi^{k_4}\rangle_{0,3}$

Exceptional genus zero case:
$\langle u,v,w\rangle_{0,3,0} = (u,v,w)$

Descendant Potential and the J-function

Let $(\mathcal{H},\Omega)$ be the symplectic loop space of $H^*(X)$ with the Poincare pairing.

Definition. *The genus g descendant potential $\mathcal{F}_g$ is a formal function on $\mathcal{H}_+$ at $\mathbf{t} = 0$ (a power series in the coefficients of $\mathbf{t}$) given by*

$$\mathcal{F}_g(\mathbf{t}) = \sum_{n\geq 0}\frac{1}{n!}\langle\mathbf{t}(\psi),...,\mathbf{t}(\psi)\rangle_{g,n}$$

$\mathcal{H}_+$ and $\mathcal{H}_-$ are complementary Lagrangian subspaces so we may identify $\mathcal{H}$ with $T^*\mathcal{H}_+$. We use this identification to define the formal function $\mathcal{J} : \mathcal{H}^+ \to \mathcal{H}$ by

$$\mathcal{J}(\mathbf{t}) = (\mathbf{t} - z, d\mathcal{F}_0(\mathbf{t}))$$

$d\mathcal{F}$ is a closed 1-form on $\mathcal{H}_+$ thus the image of J is a Lagrangian submanifold. Let us call this Lagrangian submanifold $\mathcal{L}$. The shift $-z$ is called the dilaton shift, and we introduce it because it makes the singularity of the cone lie at the origin.

Ancestors, and Proof that $\mathcal{L}$ is Overruled

To prove that $\mathcal{L}$ is an overruled cone we must introduce a broader class of invariants. There is a morphism $ct : X_{g,m+l,d} \to M_{g,m}$ called the contraction map given by forgetting l marked points and the map to X. Components that become unstable contract to points. Let $\bar{L}_i$ be the pullback of the line bundle L_i on $M_{g,m}$ by ct and let $\bar{\psi}_i$ be the first Chern class of $\bar{L}_i$. We introduce new correlator notations to denote intersection numbers including the classes $\bar{\psi}_i$. These correlators depend on a parameter $\tau \in H^*(X)$ which we associate to the marked points forgotten by ct which don't have ancestor classes.

$$\langle \mathbf{a}_1(\psi,\bar{\psi}),...,\mathbf{a}_m(\psi,\bar{\psi})\rangle^{\tau}_{g,m,d} := \sum_l \frac{1}{l!}\int_{[X_{g,m+l,d}]} \prod_{i=1}^{m}(ev_i^*\mathbf{a}_i)(\psi_i,\bar{\psi}_i) \prod_{i=m+1}^{m+l} ev_i^*\tau$$

$$\langle \mathbf{a}_1(\psi,\bar{\psi}),...,\mathbf{a}_m(\psi,\bar{\psi})\rangle^{\tau}_{g,m} := \sum_d Q^d \langle \mathbf{a}_1(\psi,\bar{\psi}),...,\mathbf{a}_m(\psi,\bar{\psi})\rangle^{\tau}_{g,m,d}$$

We also introduce a new generating function

$$\bar{\mathcal{F}}^{\tau}_g(\mathbf{t}) = \sum \frac{1}{m!}\langle \mathbf{t}(\bar{\psi}),...,\mathbf{t}(\bar{\psi})\rangle^{\tau}_{g,m}$$

called the ancestor potential. By definition, the terms with $(g,m) \in \{(0,0),(0,1),(0,2),(1,0)\}$ are zero because they correspond to unstable curve types so the corresponding spaces $M_{g,m}$ don't exist.

Definition. *Let S_τ be the matrix with z^{-1} series coefficients given by*

$$(S_\tau u,v) = (u,v) + \sum_{i=0}^{\infty} z^{-1-i}\langle u\psi^i, v\rangle^{\tau}_{0,2}$$

where $u,v \in H(X)$.

This will be the matrix that maps $\mathcal{L}$ to $\mathcal{L}_\tau$. To prove this we will first prove a "quantum" version that involves all genera, then take the "classical limit". The quantum version of the Lagrangian section $\mathcal{L}$ is

$$\mathcal{D} = \exp(\sum_{g=0}^{\infty} \hbar^{g-1}\mathcal{F}^{\tau *}_g)$$

considered as a function on $\mathcal{H}_+$ depending on $\hbar$. Here $*$ denotes the dilaton shift: $\mathcal{F}(\mathbf{t}) = \mathcal{F}^*(\mathbf{t}-z)$. $\mathcal{F}^*$ is a formal function in a neighborhood of $-z$. Functions and operators that are power series of $\hbar$ are called asymptotic, so $\mathcal{D}$ is an asymptotic function of $\mathcal{H}^+$. $\mathcal{D}^\tau$ is similarly defined for the ancestor potentials. In order to quantize S_τ, it must preserve the symplectic form Ω. Let † denote the adjoint with respect to Ω. If M is a matrix over $R((z))$ than $M^\dagger(z) = M^*(-z)$, where * denotes the adjoint with respect to the Poincare pairing.

Lemma. $S_\tau^\dagger S_\tau = I$

Proof. The proof is based on the identity

$$\sum_\alpha \langle \psi^k u, 1, \phi_\alpha \rangle_{0,2}^\tau \langle \phi^\alpha, 1, \psi^l v \rangle_{0,2}^\tau = \langle \psi^k u, 1, \psi^l v \rangle_{0,2}^\tau$$

which is the WDVV equation for ancestors with two inputs set to 1. The proof is left as an exercise. By the string equation, S can be written as follows.

$$(Su, v) = \sum_k z^{-1} \langle \psi^k u, 1, v \rangle$$

Now

$$\begin{aligned}
(S_\tau^\dagger S_\tau u, v) &= \sum_\alpha (S_\tau u, \phi_\alpha)(S_\tau^\dagger \phi^\alpha, v) \\
&= \sum_{k,l,\alpha} (-1)^l z^{-k-l} \langle \psi^k u, 1, \phi_\alpha \rangle \langle \phi^\alpha, 1, \psi^l v \rangle \\
&= \sum_{k,l,\alpha} (-1)^l z^{-k-l} \langle \psi^k u, 1, \psi^l v \rangle \\
&= \sum_{k,l,\alpha} (-1)^l z^{-k-l} (\langle \phi^{k-1} u, \phi^l v \rangle + \langle \phi^k u, \phi^{l-1} v \rangle) \\
&= (u, v)
\end{aligned}$$

$\square$

According to the appendix, the quantization of S_τ^{-1} acts by

$$\widehat{S^{-1}} G(\mathbf{q}) = e^{\Omega(\mathbf{q}, S^{-1}\pi_+ S\mathbf{q})} G(\pi_+ S\mathbf{q})$$

The actual operator sending $\mathcal{D}^\tau$ to $\mathcal{D}$ differs from this by a constant factor $F(\tau)$ but this doesn't change the classical limit.

Theorem.

$$e^{F_1(\tau)} \widehat{S_\tau^{-1}} \mathcal{D}^\tau = \mathcal{D}$$

Proof. As a midway point between ancestors and descendants we introduce, for lack of a better name, the τ dependent descendant potential:

$$\mathcal{F}_g^\tau(\mathbf{t}) = \sum \frac{1}{m!} \langle \mathbf{t}(\psi), ..., \mathbf{t}(\psi) \rangle_{g,m}^\tau$$

Lemma. *When the correlators exist, that is $(g, m) \notin \{(0,0), (0,1), (0,2), (1,0)\}$, we have*

$$\langle \mathbf{t}(\psi), ..., \mathbf{t}(\psi) \rangle_{g,m}^\tau = \langle [S_\tau \mathbf{t}]_+(\bar{\psi}), ..., [S_\tau \mathbf{t}]_+(\bar{\psi}) \rangle_{g,m}^\tau$$

Where $[\,]_+$ denotes the power series truncation of a Laurent series. It follows that for $g > 1$ we have $\mathcal{F}_g^\tau(\mathbf{t}) = \bar{\mathcal{F}}_g^\tau([S_\tau \mathbf{t}]_+)$.

Proof. To write descendants in terms of ancestors, we must find the discrepancy between L_i and $\bar{L}_i$. ct induces a section of $Hom(\bar{L}_i, L_i)$ that is non vanishing on the compliment of the divisor D of curves where x_i lies on a component collapsed by ct. It turns out that the normal bundle of D is identified with $Hom(\bar{L}, L)$ so we have $[D] = \psi - \bar{\psi}$. D is the union of the images of the maps

$$X_{0,2+l'',d''} \times_X X_{g,1+l'+m,d'} \to X_{m,l,d}$$

over all l', l'', d', d'' such that $l' + l'' = l$ and $d' + d'' = d$. Here $\times_X$ denotes taking the preimage of ΔX under $ev_2 \times ev_1$.

$$\langle u\psi^a \bar{\psi}^b, \ldots \rangle^\tau_{g,m} = \langle u\psi^{a-1}\bar{\psi}^{b+1}, \ldots \rangle^\tau_{g,m} + \langle u\psi^{a-1}, \phi_\alpha \rangle^\tau_{0,2} \langle \phi^\alpha \bar{\psi}^b, \ldots \rangle^\tau_{g,m}$$

Applying this formula iteratively we get

$$\langle u\psi^a, \ldots \rangle^\tau_{g,m} = \langle u\bar{\psi}^a, \ldots \rangle^\tau_{g,m} + \sum_{i=0}^{a-1} \langle u\psi^i, \phi_\alpha \rangle^\tau_{0,2} \langle \phi^\alpha \bar{\psi}^{a-1-i}, \ldots \rangle^\tau_{g,m}$$

□

Lemma. $\mathcal{F}(\mathbf{t}) = \mathcal{F}^\tau(\mathbf{t} - \tau)$

Proof. To relate $\mathcal{F}$ to $\mathcal{F}^\tau$ we do a Taylor series expansion.

$$\mathcal{F}(\mathbf{t} + \epsilon\tau) = \sum \frac{\epsilon^n}{n!} \frac{d^n}{d^n\epsilon} \mathcal{F}(\mathbf{t} + \epsilon\tau) \Big|_{\epsilon=0}$$

$$\frac{d^n}{d^n\epsilon}\mathcal{F}(\mathbf{t}+\epsilon\tau)\Big|_{\epsilon=0} = \sum_{m=0}^{\infty} \frac{1}{m!}\frac{d^n}{d^n\epsilon}\langle \mathbf{t}(\psi)+\epsilon\tau, \ldots, \mathbf{t}(\psi)+\epsilon\tau\rangle\Big|_{\epsilon=0} = \sum_{m=0}^{\infty} \frac{1}{(m-n)!}\langle \underbrace{\mathbf{t}(\psi), \ldots, \mathbf{t}(\psi)}_{m-n}, \underbrace{\tau, \ldots, \tau}_{n}\rangle$$

When $m < n$ the summand vanishes. Letting $k = m - n$ we have

$$\mathcal{F}(\mathbf{t} + \tau) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{1}{n!k!}\langle \underbrace{\mathbf{t}(\psi), \ldots, \mathbf{t}(\psi)}_{k}, \underbrace{\tau, \ldots, \tau}_{n}\rangle = \sum_{k=0}^{\infty} \frac{1}{k!}\langle \mathbf{t}(\psi), \ldots, \mathbf{t}(\psi)\rangle_\tau = \mathcal{F}^\tau(\mathbf{t})$$

Replacing $\mathbf{t}$ with $\mathbf{t} - \tau$ gives the result. □

Combining the previous two lemmas we have for $g > 1$

$$\mathcal{F}_g(\mathbf{t}) = \bar{\mathcal{F}}^\tau_g([S_\tau(\mathbf{t} - \tau)]_+)$$

The dilaton shift simplifies this formula. By convention we let $\mathbf{q} = \mathbf{t} - z$ so that $\mathcal{F}(\mathbf{t}) = \mathcal{F}^*(\mathbf{q})$.

Lemma. $\mathcal{F}^*_g(\mathbf{q}) = \mathcal{F}^{\tau *}_g([S_\tau \mathbf{q}]_+)$

Proof. First let us calculate $[S_\tau z]_+$. Let $v \in H^*(X)$

$$([S_\tau z]_+, v) = (z, v) + \left[\sum_{i=0}^{\infty} z^{-1-i} \langle z\psi^i, v\rangle_{0,2}^{\tau}\right]_+$$

$$= (z,v) + \langle 1, v\rangle_{0,2}^{\tau}$$

$$= (z,v) + \langle 1, v, \tau\rangle_{0,3,0}$$

$$= (z,v) + (v,\tau)$$

Thus $[S_\tau z]_+ = z + \tau$. Now we see that

$$[S_\tau \mathbf{q}]_+ = [S_\tau(\mathbf{t} - z)]_+ = [S_\tau \mathbf{t}]_+ - [S_\tau z]_+ = [S_\tau \mathbf{t}]_+ - \tau - z = [S_\tau \mathbf{t} - \tau]_+ - z$$

This proves the lemma. $\square$

We must find the discrepancy of our formula in genus 0 and genus 1. In genus zero there are three "missing" ancestors so we get three terms in the discrepancy:

$$\mathcal{F}_0(\mathbf{t}) - \bar{\mathcal{F}}_0^\tau([S_\tau(\mathbf{t}-\tau)]_+) = \mathcal{F}_0^\tau(\mathbf{t}-\tau) - \bar{\mathcal{F}}_0^\tau([S_\tau(\mathbf{t}-\tau)]_+)$$

$$= \langle\rangle_{0,0}^\tau + \langle \mathbf{t}(\psi) - \tau\rangle_{1,0}^\tau + \langle \mathbf{t}(\psi) - \tau, \mathbf{t}(\psi) - \tau\rangle_{2,0}^\tau$$

We can use the dilaton equation to simplify. The dilaton equation implies the following for τ dependent correlators.

$$\langle \psi, ...\rangle_{0,n+1}^\tau = \langle ..., \tau\rangle^\tau + (n-2)\langle ...\rangle^\tau$$

This can be used to show that the genus zero discrepancy is equal to $\langle \mathbf{q}(\psi), \mathbf{q}(\psi)\rangle_{0,2}^\tau$. In genus 1, there is one discrepancy term, $\langle\rangle_{1,0}^\tau$, which we denote $F_1(\tau)$ and is called the genus 1 Gromov-Witten potential. Putting everything together, we have shown

$$\mathcal{D}(\mathbf{q}) = e^{F_1(\tau)} e^{\langle \mathbf{q}(\psi), \mathbf{q}(\psi)\rangle_{0,2}^\tau / 2\hbar} \mathcal{D}^\tau([S_\tau \mathbf{q}]_+)$$

It remains to check that $\langle \mathbf{q}(\psi), \mathbf{q}(\psi)\rangle_{0,2}^\tau = \Omega(\mathbf{q}, S^\dagger \pi_+ S\mathbf{q})$. The proof is very similar to the proof that $S_\tau^\dagger S = I$ and it is left as an exercise. $\square$

Corollary. $S_\tau \mathcal{L} = \mathcal{L}_\tau$

Theorem 5. *$\mathcal{L}$ is an overruled Lagrangian cone.*

We split this into the two following lemmas. Intuitively the S_τ roll $\mathcal{L}$ against $\mathcal{H}_+$ proving its overruledness.

Lemma. *$\mathcal{L}^\tau$ is tangent to $\mathcal{H}_+$ along $z\mathcal{H}_+$.*

Proof. The dimension of $M_{0,m}$ is $m-3$ so if $\mathbf{t}_1, ..., \mathbf{t}_m \in \mathcal{H}_+$ are power series with at least $m-2$ of them in $z\mathcal{H}_+$ than $\langle \mathbf{t}_1, ..., \mathbf{t}_m\rangle_{0,m,d}^\tau = 0$. This means that all first and second derivatives of $\mathcal{F}_g^\tau$ vanish along $z\mathcal{H}_+$. This means that $\mathcal{L}^\tau$ is tangent to $\mathcal{H}_+$ along $z\mathcal{H}_+$. $\square$

Lemma. *For any $\mathbf{f} = J(\mathbf{q})$ in a formal neighborhood of $J(z\mathbf{1})$ there exists τ such that $S_\tau \mathbf{f} \in z\mathcal{H}_+$*

Proof. Let $\mathbf{f} = \mathbf{q} + \mathbf{p}$ be a point in $\mathcal{L}$ where $\mathbf{q} \in \mathcal{H}_+$ and $\mathbf{p} \in \mathcal{H}_-$. If $\pi_+ S_\tau \mathbf{q} \in z\mathcal{H}_+$ than $\pi_+ S_\tau \mathbf{f} \in z\mathcal{H}_+$ because S_τ fixes $\mathcal{H}_-$. If $\pi_+ S_\tau \mathbf{f} \in z\mathcal{H}_+$ than $S_\tau \mathbf{f} \in z\mathcal{H}_+$ because $S_\tau \mathbf{f}$ is guaranteed to lie in $\mathcal{L}$ which contains $z\mathcal{H}_+$. Therefore, $S_\tau \mathbf{f} \in z\mathcal{H}_+$ is equivalent to $\pi_+ S_\tau \mathbf{q} \in z\mathcal{H}_+$. This is equivalent to the z^0 term of $(S_\tau \mathbf{q}, v)$ vanishing for all $v \in H^*(X)$. This z^0 term is $\langle \mathbf{q}(\psi), 1, v\rangle_{0,3}^\tau$. Vanishing of $\langle \mathbf{q}(\psi), 1, v\rangle_{0,3}^\tau$ for all v is equivalent to τ being a critical point of $\langle \mathbf{q}(\psi), 1\rangle_{0,2}^\tau$ considered as a function of τ. When $\mathbf{q} = q_0 - z$, it is the following function modulo $Q_1, ..., Q_r$.

$$\langle q_0 - \psi, 1\rangle_{0,2,0}^\tau = (q_0, \tau) - \langle 1\rangle_{0,2,0}^\tau = (q_0, \tau) - \frac{1}{2}(\tau, \tau)$$

This function has a unique non-degenerate critical point $\tau = q_0$ therefore we can solve for a critical point if we reintroduce the terms with $Q_1, ..., Q_r$. This guarantees existence of a unique critical point $\tau(\mathbf{q})$ in a formal neighborhood of $\mathbf{q} = -z$. ☐

Now we even have an explicit parameterization of $\mathcal{L}$ as the union of the ruling spaces $zT_\tau = zS_\tau \mathcal{H}_+$.

The $\mathcal{D}$-module property

Let $\mathcal{D}$ be the associative algebra over $\mathbb{C}[z]$ generated by $Q_1, ..., Q_r$ and $zQ_1\partial_{Q_1}, ..., zQ_r\partial_{Q_r}$ with the relations $[zQ_i\partial_{Q_i}, Q_j] = \delta_{ij} zQ_j$. Let $d_1, ..., d_r$ be a basis of $H_2(X, \mathbb{Z})$ consisting of classes representable by holomorphic curves, and let $p_1, ..., p_r$ be a Poincare dual basis of $H^2(X)$. We define an action of $\mathcal{D}$ on $\mathcal{H}$ where Q_i acts as a multiplication operator and $zQ_i\partial_{Q_i}$ acts by $zQ_i\partial_{Q_i} - p_i$.

Definition. *An overruled Lagrangian cone $\mathcal{L}$ is said to satisfy the $\mathcal{D}$-module property if it satisfies any of the following equivalent conditions:*

1. *Tangent spaces of $\mathcal{L}$ are $\mathcal{D}$ modules*
2. *Ruling spaces of $\mathcal{L}$ are $\mathcal{D}$ modules*
3. *The vector field $f \mapsto z^{-1}Df$ is tangent to $\mathcal{L}$ for all $D \in \mathcal{D}$.*

The first two conditions are equivalent because $\mathcal{D}$ commutes with z. f/z being in a tangent space is equivalent to f being in the corresponding ruling space so 2 and 3 are equivalent. Note that $z^{-1}\mathcal{D}$ is not closed as an associative algebra, but it is closed as a lie algebra, so it makes sense for it to act infinitesimally on $\mathcal{H}$.

Theorem 6. *$\mathcal{L}$ satisfies the $\mathcal{D}$ module property.*

Proof. The divisor equation implies that S_τ satisfies a system of differential equations. Let $\{\phi_\alpha\}$ be a basis of $H^*(X)$ with $\phi_0 = 1$ and $\phi_i = p_i$ for $1 \le i \le r$. Let τ_α be the coefficients of τ. Let us compute the terms of $\partial_{\tau_i}(S_\tau u, v)$.

$$\partial_{\tau_i}\langle u\psi^k, 1, v\rangle_{0,3,d}^\tau = \langle u\psi^k, 1, v, \phi_i\rangle_{0,4,d}^\tau = (\phi_i, d)\langle u\psi^k, 1, v\rangle_{0,3,d}^\tau + \langle \phi_i u\psi^{k-1}, 1, v\rangle_{0,3,d}^\tau$$

Where the second term vanishes if $k = 0$. In the string equation case $i = 0$, this becomes

$$\partial_{\tau_0} \langle u\psi^k, 1, v\rangle^{\tau}_{0,3} = \langle u\psi^{k-1}, 1, v\rangle^{\tau}_{0,3}$$

In the case $1 \le i \le r$ it becomes

$$\partial_{\tau_i} \langle u\psi^k, 1, v\rangle^{\tau}_{0,3} = Q_i \partial_{Q_i} \langle u\psi^k, 1, v\rangle^{\tau}_{0,3} + \langle \phi_i u\psi^{k-1}, 1, v\rangle^{\tau}_{0,3}$$

These are equivalent to the following differential equations.

$$z\partial_0 S_\tau = S_\tau \qquad z\partial_{\tau_i} S_\tau = zQ_i\partial_{Q_i} S_\tau + S_\tau p_i$$

If we take the adjoint with respect Ω of the second equation we get

$$-z\partial_{\tau_i} S^\dagger_\tau = -zQ_i\partial_{Q_i} S^\dagger_\tau + p_i S^\dagger_\tau$$

Since $S^{-1} = S^\dagger$ this is equivalent to

$$(zQ_i\partial_{Q_i} - p_i)S^{-1} = z\partial_{\tau_i} S^{-1}$$

Let $S^{-1}_\tau h$, with $h \in z\mathcal{H}_+$, be a point in the ruling space zT_τ.

$$(zQ_i\partial_{Q_i} - p_i)(S^{-1}h) = z(\partial_{\tau_i} S^{-1})h + \sum_\alpha (Q_i\partial_{Q_i}\tau_\alpha) z\partial_{\tau_\alpha} S^{-1}_\tau h + zS^{-1}(Q_i\partial_{Q_i} h)$$

All of these terms lie in zT_τ so zT_τ is a $\mathcal{D}$ module. □

Part 2: Incarnations

Reconstruction

Reconstruction is the fact, first stated in this form in [1], that an overruled Lagrangian Cone with the $\mathcal{D}$ module property is completely determined by a single point on it. Let $\mathcal{L}$ be an overruled Lagrangian cone with the $\mathcal{D}$ module property. The last form of the $\mathcal{D}$-module property implies that if $\mathbf{f} \in \mathcal{L}$, than $\exp(z^{-1}D)f \in \mathcal{L}$, whenever the exponential exists. Unfortunately the exponential does not always exist. For example $\exp(z^{-1})$ does not lie in $\mathcal{H}$. Still, we can say that $\exp(\epsilon z^{-1}D)f$ maps $\mathrm{spec}(\mathbb{C}[[\epsilon]])$ onto $\mathcal{L}$. If $z^{-1}\mathcal{D}f$ spans the tangent space of $\mathcal{L}$ at $\mathbf{f}$, than $D \mapsto exp(z^{-1}D)f$ maps a formal neighborhood of 0 in $z^{-1}\mathcal{D}$ surjectively onto a formal neighborhood of $\mathbf{f}$ in $\mathcal{L}$.

Lemma. *If $p_1, ..., p_r$ generate $H^*(X)$, and $\mathbf{f}$ is the point of $\mathcal{L}$ that maps to $-z$ under π_+, than $z^{-1}\mathcal{D}f$ spans $T_{\mathbf{f}}\mathcal{L}$.*

Proof. Let $\Phi_\alpha(p_1, ..., p_r)$ be polynomials that form a basis of $H^*(X)$. The vectors $v_\alpha = z^{-1}\Phi_\alpha(P_1, ..., P_r)f$ form a basis of $T_{\mathbf{f}}$ over $R[[z]]$. Since $T_{\mathbf{f}}$ projects isomorphically to $\mathcal{H}_+$, it is sufficient to show that $\pi_+ v_\alpha$ form a basis of $\mathcal{H}_+$. $\pi_+ v_\alpha$ do form a basis modulo the maximal ideal I_R of R.

$$\pi_+[z^{-1}\Phi_\alpha(P_1, ..., P_r)f] = -\Phi_\alpha(-p_1, ..., -p_r) \mod I_R$$

and this implies that they form a basis in general. □

In the case of Gromov Witten theory, reconstruction means that all Gromov Witten invariants can be extracted from the single point $\mathbf{f} = \mathcal{J}(-z)$. The original goal of this research was to determine for which $\mathbf{f} \in \pi_+^{-1}(-1z)$ we get an overruled Lagrangian cone. This would form a nice description of the space of overruled Lagrangian Cones. It turns out that $\mathbf{f}$ must satisfy a system of differential equations corresponding to relations in the cohomology algebra $H^*(X)$, otherwise the resulting tangent space will be too big and won't project isomorphically to $\mathcal{H}_+$. However, it is hard to tell from $\mathbf{f}$ alone, or the differential equations that $\mathbf{f}$ satisfies, weather the resulting overruled cone is Lagrangian. For this reason we took a different approach.

An overruled cone $\mathcal{L} \subset \mathcal{H}$ is determined by its tangent spaces. $\mathcal{L}$ can thus be regarded as a submanifold in the Grassmannian of $R[[z]]$ submodules of $\mathcal{H}$ that project isomorphically to $\mathcal{H}_+$. Furthermore, if $\mathcal{L}$ has the $\mathcal{D}$ module property than it can be regarded as a submanifold in the Grassmannian of $\mathcal{D}$-modules. If $\mathcal{L}$ is Lagrangian than it is restricted to yet a smaller Grassmannian. I will refer to these Grassmannians as $\mathcal{G}, \mathcal{G}_{lag}, \mathcal{G}_{\mathcal{D}}$, and $\mathcal{G}_{lag,\mathcal{D}}$. The set of subspaces projecting isomorphically to $\mathcal{H}_+$ is an isolated connected component of the Grassmannian of $\mathbb{Q}[[z]]$ modules, see [6]. Here, Grassmannian will always refer to the subset in this component. The first goal of this section will be to give models of these Grassmannians. The second goal will be to see how $z^{-1}\mathcal{D}$ acts. It turns out that the action of $z^{-1}\mathcal{D}$ preserves $\mathcal{G}$, $\mathcal{G}_{\mathcal{D}}$ and $\mathcal{G}_{lag,\mathcal{D}}$. In the case of $\mathcal{G}_{\mathcal{D}}$ and $\mathcal{G}_{lag,\mathcal{D}}$, $\mathcal{D}$ acts trivially by definition, so the action descends to an action of $\tilde{\mathcal{D}} = z^{-1}\mathcal{D} \bmod \mathcal{D}$ which is isomorphic to a subset of the lie algebra of vector fields on spec R.

Definition. *Let us fix notation for some groups acting on $\mathcal{H}$.*

- *$LGL(n,R)$ is the group $GL(n, R((z)))$. It is the loop group corresponding to $GL(n,R)$.*
- *$L^{(2)}GL(n,R)$ is the subgroup of $LGL(n,R)$ that preserves Ω. Explicitly $L^{(2)}GL(n,R) = \{S \in GL(n,R) : S^\dagger S = I\}$ where $S^\dagger(z) = S^*(-z)$.*
- *$LGL(n,R)_+$ and $L^{(2)}GL(n,R)_+$ are the subgroups that preserve $\mathcal{H}_+$. I will call these the upper loop group and upper twisted loop group. Elements of these have the form $M_0 + zM_1 + z^2M_2 + ...$ where M_0 is invertible.*
- *$LGL(n,R)_-$ and $L^{(2)}GL(n,R)_-$ are the subgroups that preserve $\mathcal{H}_-$. I will call these the lower loop group and lower twisted loop group. Elements of these have the form $I + zM_1 + z^2M_2 + ...$*

$R[[z]]$ Modules and the Lower Loop Group

We will specify an element $T \in \mathcal{G}$ by the unique element of $LGL(n,R)_-$ that maps $\mathcal{H}_+$ to T. It is given by the matrix who's columns are the vectors $\pi_+^{-1}(\phi_\alpha) \cap T$. In this way $\mathcal{G}$ is identified with $LGL(n,R)_-$.

Lemma. *Let $T \in \mathcal{G}$, and let $M \in LGL(n,R)$ be the element of $LGL(n,R)_-$ that maps $\mathcal{H}_+$ to T. T is Lagrangian if and only if $M \in L^{(2)}GL(n,R)$.*

Proof. If $M \in L^{(2)}GL(n,R)$ than it must send Lagrangians to Lagrangians so T is Lagrangian. Now suppose T is Lagrangian. This means that for all $f,g \in \mathcal{H}_+$,

$$\Omega(Mf,Mg) = \Omega(MM^\dagger g,f) = 0$$

Since $\mathcal{H}_+$ is Lagrangian, we must have

$$MM^\dagger \mathcal{H}_+ \subset \mathcal{H}_+$$

This means $MM^\dagger$ must be a power series in z but by construction M and thus $MM^\dagger$ is a power series in z^{-1} with constant term I thus $MM^\dagger$ is just I. □

$\mathcal{D}$ Modules and Connections

As before, let $T \in \mathcal{G}$, and let $M \in LGL(n,R)$ be the element of $LGL(n,R)_-$ that maps $\mathcal{H}_+$ to T. We must find a condition under which the operators $zQ_i\partial_{Q_i} - p_i$ preserve T. This is equivalent to $M^{-1}(zQ_i\partial_{Q_i} - p_i)M$ preserving $\mathcal{H}_+$. Expanding gives

$$M^{-1}(zQ_i\partial_{Q_i} - p_i)M = zQ_i\partial_{Q_i} + M^{-1}(zQ_i\partial_{Q_i}M) - M^{-1}p_iM$$

So T is a $\mathcal{D}$ module if and only if $M^{-1}(zQ_i\partial_{Q_i}M) - M^{-1}p_iM$ has no z^{-1} terms. Note that $zQ_i\partial_{Q_i}M$ is divisible by z so $M^{-1}(zQ_i\partial_{Q_i}M) - M^{-1}p_iM$ can not have z terms. It follows that if T is a $\mathcal{D}$ module, $M^{-1}(zQ_i\partial_{Q_i}M) - M^{-1}p_iM$ is constant in z.Unfortunately such M do not form a subgroup.

In general, a mapping of the elements $zQ_i\partial_{Q_i} \in \mathcal{D}$ to differential operators of the form $zQ_i\partial_{Q_i} + A_i$ where A_i are matrices in $End(\mathcal{H}_+, R[[z]])$ is called a connection. I will refer to the connection itself as $zQ_i\partial_{Q_i} + A_i$. As exemplified above, $LGL(n,R)$ acts on connections by conjugation which is usually called gauge transformation. We will call connections that can be obtained from $zQ_i\partial_{Q_i} - p_i$ by a gauge transformation trivializable. Let be Γ the set of trivializable connections $zQ_i\partial_{Q_i} + A_i$ where A_i are constant in z. We have shown that there is a map $\mathcal{G}_{\mathcal{D}} \to \Gamma$. It will turn out to be a bijection but let us first describe Γ a bit.

Lemma. *If $zQ_i\partial_{Q_i} + A_i$ is trivializable than $[A_i, A_j] = 0$ and $Q_j\partial_{Q_j}A_i - Q_i\partial_{Q_i}A_j = 0$ for all i,j.*

Proof. Suppose there exists $M \in LGL(n,R)$ such that $M^{-1}(zQ_i\partial_{Q_i} - p_i)M = zQ_i\partial_{Q_i} + A_i$. This means

$$zQ_i\partial_{Q_i} - M^{-1}p_iM + zM^{-1}Q_i\partial_{Q_i}M = zQ_1\partial_{Q_1} + A_i$$

$$-p_iM + zQ_i\partial_{Q_i}M = MA_i$$

$$Q_i\partial_{Q_i}M = z^{-1}(MA_i + p_iM)$$

The fact that the differential operators $Q_i\partial_{Q_i}$ and $Q_j\partial_{Q_j}$ commute will impose constraints on A_i and A_j.

$$Q_i\partial_{Q_i}(Q_j\partial_{Q_j}M) = Q_i\partial_{Q_i}(z^{-1}(MA_j + p_jM))$$

$$= z^{-1}((Q_i\partial_{Q_i}M)A_j + M(Q_i\partial_{Q_i}A_j) + p_jQ_i\partial_{Q_i}M)$$

$$= z^{-1}(z^{-1}(MA_i + p_iM)A_j + M(Q_i\partial_{Q_i}A_j) + p_jz^{-1}(MA_i + p_iM))$$
$$= z^{-2}(MA_iA_j + p_iMA_j + p_jMA_i + p_jp_iM) + z^{-1}M(Q_i\partial_{Q_i}A_j)$$

Now subtract from this with the result of applying $Q_i\partial_{Q_i}$ then $Q_j\partial_{Q_j}$.

$$Q_i\partial_{Q_i}(Q_j\partial_{Q_j}M)M - Q_j\partial_{Q_j}(Q_i\partial_{Q_i}M)M = z^{-2}M[A_i, A_j] + z^{-1}M(Q_i\partial_{Q_i}A_j - Q_j\partial_{Q_j}A_i) = 0$$

□

If A_i are constant in z and equal to p_i mod Q than there is a converse to this theorem.

Lemma. *Suppose $zQ_i\partial_{Q_i} + A_i$ is a connection satisfying $[A_i, A_j] = 0$ and $Q_j\partial_{Q_j}A_i - Q_i\partial_{Q_i}A_j = 0$ with A_i constant in z and equal to p_i mod Q, than it is trivializable by an element of $LGL(n, R)_-$.*

Proof. We must find the gauge transformation M turning $zQ_i\partial_{Q_i} - p_i$ into $zQ_i\partial_{Q_i} + A_i$. As before, this can be written

$$Q_i\partial_{Q_i}M = z^{-1}(MA_i + p_iM)$$

If we assume that $M = I + z^{-1}M_1 + z^{-2}M_2 + ...$ than we may rewrite the equation as a recurrence relation.

$$Q_i\partial_{Q_i}M_{k+1} = M_kA_i + p_iM_k$$

These relations are very redundant. The $Q_1^{n_1} \cdots Q_r^{n_r}$ term of M_{k+1} can be computed using the recurrence relation for any i such that n_i is not zero. However, our assumptions about A_i guarantee that each gives the same result. The sequence of matrices M_r must eventually be divisible by higher and higher degree monomials of Q's, because the constant terms of A_i and p_i are nilpotent, thus M is defined over $R((z))$. □

Combining our results, we have proved that $\mathcal{G}_\mathcal{D}$ is in bijection with Γ which is in bijection with the set of tuples of r commuting matrices over R, equal to p_i mod Q, that form a closed 1-form.

Remark. A tuple of r commuting matrices over R, equal to p_i mod Q is a formal deformations of $H^*(X)$ over $Spec(\mathbb{C}[[Q_1, ..., Q_r]])$. In Gromov-Witten theory the connection corresponding S_0 is the much studied quantum cohomology.

Lagrangian $\mathcal{D}$ Modules and Symmetric Connections

Theorem 7. *A $\mathcal{D}$ module $T \in \Gamma$ is Lagrangian if and only if the corresponding connection consists of matrices self adjoint with respect to the Poincare pairing.*

Proof. First we show that if M is a gauge transformation such that $M^{-1}(Q_i\partial_{Q_i})M = Q_i\partial_{Q_i} + A_i$, than A_i is self adjoint if and only if M is in the twisted loop group. The constant term of $M^\dagger M$ is I so it suffices to show that the derivative of $Q_i\partial_{Q_i}(M^\dagger M)$ is zero. First calculate

$$Q_i\partial_{Q_i}M = z^{-1}MA_i$$

$$Q_i\partial_{Q_i}M^\dagger = -z^{-1}M^\dagger A_i^\dagger$$

Then use these to calculate

$$
\begin{aligned}
Q_i\partial_{Q_i}(M^\dagger M) &= (Q_i\partial_{Q_i}M^\dagger)M + M^\dagger(Q_i\partial_{Q_i}M) \\
&= -z^{-1}M^\dagger A^\dagger M + z^{-1}M^\dagger AM \\
&= z^{-1}M^\dagger(A - A^\dagger)M
\end{aligned}
$$

This vanishes if and only if $A - A^\dagger = 0$. If $\mathcal{D}$ acts on $\mathcal{H}$ with the connection $Q_i\partial_{Q_i} - p_i$ than the connection corresponding to a subspace $M\mathcal{H}_+$ is $M^{-1}(Q_i\partial_{Q_i} - p_i)M$. Since

$$Q_i\partial_{Q_i} - p_i = e^{p\log Q/z}(Q_i\partial_{Q_i})e^{-p\log Q/z}$$

we see that $e^{-p\log Q/z}M$ is the gauge transformation mapping $Q_i\partial_{Q_i}$ to our connection $Q_i\partial_{Q_i} + A_i$. $e^{-p\log Q/z}MM^\dagger e^{p\log Q/z}$ is identity if and only if $MM^\dagger$ is identity so the result follows. Technically $\log Q$ is not in our ring but we can simply extend R to $R[\log Q]$. Note that we don't need power series in $\log Q$ because p_i are nilpotent. □

Action of $\tilde{\mathcal{D}}$

It will be advantageous to change our perspective. If $\mathcal{L}$ has the $\mathcal{D}$ module property with respect to a connection, than $S\mathcal{L}$ has the $\mathcal{D}$ module property with respect to the gauge transformed connection. Suppose $M\mathcal{H}_+$ is a tangent space of $\mathcal{L}$ corresponding to the connection $zQ_i\partial_{Q_i} + A_i$. $S^{-1}\mathcal{L}$ is an overruled Lagrangian cone tangent to $\mathcal{H}_+$, with the $\mathcal{D}$-module property with respect to $zQ_i\partial_{Q_i} + A_i$. Let $D \in \mathcal{D}$ and let D_A denote its action on $\mathcal{H}$ given by the connection $zQ_i\partial_{Q_i} + A_i$. By the reconstruction theorem, the family of matrices $e^{\epsilon D_A}I$ map $\mathcal{H}_+$ to a family of tangent spaces to $S^{-1}\mathcal{L}$. We do a Birkoff factorization $e^{\epsilon z^{-1}D_A}I = U_\epsilon V_\epsilon$ where $U_\epsilon = U_{\epsilon,0} + z^{-1}U_{\epsilon,1} + ...$ and $V_\epsilon = I + zV_{\epsilon,1} + ...$ Note that $U_0 = V_0 = I$. The derivative at $\epsilon = 0$ is

$$z^{-1}D_AI = U' + V'$$

Note that D_AI is a polynomial in $A_1, ..., A_r$ with coefficients in R. We want to take the derivative of the gauge transformation of A with respect to U_ϵ.

$$
\begin{aligned}
\frac{d}{dt}[U_\epsilon^{-1}(zQ_i\partial_{Q_i} - A_i)U_\epsilon]_{\epsilon=0} &= zQ_i\partial_{Q_i}U' - [U', A_i] \\
&= zQ_i\partial_{Q_i}(z^{-1}D_AI - V') - [z^{-1}D_AI - V', A_i] \\
&= Q_i\partial_{Q_i}(D_AI) - zQ_i\partial_{Q_i}V' + [V', A_i]
\end{aligned}
$$

Note that we have used the fact that A_i all commute. If $A_i = p_i + z^{-1}A_1 + ...$ than $[V', A]$ is divisible by z. The whole expression can not have any positive powers of z because neither U_ϵ nor A do. It follows that the infinitesimal change in connection that happens when you are at the connection given by the matrices A_i, and moving along the vector field $\mathbf{f} \mapsto z^{-1}D\mathbf{f}$, is $Q_i\partial_{Q_i}(D_AI)$. A polynomial in commuting symmetric matrix is a symmetric matrix so $\mathcal{D}$ indeed preserves the subspace of symplectic connections. In other words, if you start with a Lagrangian $\mathcal{D}$-module, reconstruction is guaranteed to make you a Lagrangian cone.

Appendix: Quantization of Lower Twisted Loop Group

We will first quantize quadratic Hamiltonians, then quantize elements in the lower twisted loop algebra $L^{(2)}\mathfrak{gl}(n,R)_-$, then finally exponentiate to get quantizations of elements of the lower twisted loop group $L^{(2)}GL(n,R)_-$.

Quadratic Hamiltonians

Quantization is a linear map $f \mapsto \hat{f}$ from Hamiltonians (functions) on $\mathcal{H}$ to asymptotic differential operators on $\mathcal{H}^+$ satisfying.

$$[\hat{f},\hat{g}] = \widehat{\{f,g\}} + \text{higer order terms in } \hbar$$

We will only need to quantize quadratic Hamiltonians (homogeneous degree 2 functions) so it is sufficient to define its action on degree 2 Darboux monomials. Let $\{q_\nu\}$ and $\{p_\nu\}$ be Darboux coordinates of $\mathcal{H}^+$ respecting the polarization $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$. Define

$$\widehat{q_\nu q_\mu} = \frac{1}{\hbar} q_\nu q_\mu \qquad \widehat{q_\nu p_\mu} = q_\nu \partial_{q_\mu} \qquad \widehat{p_\nu p_\mu} = \hbar \partial_{q_\nu} \partial_{q_\mu}$$

You can check that this is a quantization.

Lower Twisted Loop Algebra

Let A be an element of the lower twisted loop algebra. This means $A = I + z^{-1}A_1 + z^{-2}A_2 \ldots$, and that A infinitesimally preserves Ω. $\hat{A}$ is defined to be the quantization of the quadratic Hamiltonian that generates the vector field $\mathbf{q} \mapsto A\mathbf{q}$. I will denote this vector field by $\vec{A}$. It follows from symplectic linear algebra that the quadratic Hamiltonian is $\frac{1}{2}\Omega(A\mathbf{f},\mathbf{f})$. Indeed,

$$d(\frac{1}{2}\Omega(A\mathbf{f},\mathbf{f}))(\mathbf{f}') = \frac{1}{2}\Omega(A\mathbf{f}',\mathbf{f}) + \frac{1}{2}\Omega(A\mathbf{f},\mathbf{f}') = \Omega(A\mathbf{f},\mathbf{f}') = (i_{\vec{A}}\Omega)_{\mathbf{f}}(\mathbf{f}')$$

where we think of $\mathbf{f}'$ as a tangent vector at $\mathbf{f}$. If we write $\mathbf{f} = \mathbf{q} + \mathbf{p}$ than the quadratic Hamiltonian decomposes into two pieces.

$$\frac{1}{2}\Omega(A(\mathbf{q}+\mathbf{p}),\mathbf{q}+\mathbf{p}) = \frac{1}{2}\Omega(A\mathbf{q},\mathbf{q}) + \Omega(A\mathbf{q},\mathbf{p})$$

The first term becomes multiplication by $\frac{1}{2\hbar}\Omega(A\mathbf{q},\mathbf{q})$. In the second term, only the power series part of $A\mathbf{q}$ will pair non trivially with $\mathbf{p}$, so it is equal to $\Omega([A\mathbf{q}]_+,\mathbf{p})$. This quantizes to differentiation by the vector field $\vec{\pi_+ A}$ where π_+ is projection to $\mathcal{H}_+$.

$$\hat{A} = \frac{1}{2\hbar}\Omega(A\mathbf{q},\mathbf{q}) + \vec{\pi_+ A}$$

Lower Twisted Loop Group

Let $S = \exp(A)$ be an element of the twisted loop group with only negative powers of z. $\hat{S}$ is defined to be $\exp(\hat{A})$ but there is a formula for $\hat{S}$ directly in terms of S. $G_t = \exp(t\hat{A})G$ is determined by the differential equation

$$\frac{d}{dt}G_t = \hat{A}G_t$$

Suppose the solution has the form $G_t = (e^{W_t}G) \circ e^{-t\pi_+ A}$.

$$\frac{d}{dt}G_t = (\frac{d}{dt}W_t e^{W_t}G + \vec{\pi_+ A} \cdot (e^{W_t}G)) \circ e^{-t\pi_+ A} = ((\frac{d}{dt}W_t) \circ e^{-t\pi_+ A} + \vec{\pi_+ A})G_t$$

Comparing this to $\hat{A}G$ we see that W_t must satisfy

$$(\frac{d}{dt}W_t)(e^{-t\pi_+ A}\mathbf{q}) = \frac{1}{2\hbar}\Omega(A\mathbf{q}, \mathbf{q})$$

or equivalently, noting that $e^{-t\pi_+ A} = \pi_+ e^{-tA}$

$$\frac{d}{dt}W_t(\mathbf{q}) = \frac{1}{2\hbar}\Omega(A\pi_+ e^{-tA}\mathbf{q}, \pi_+ e^{-tA}\mathbf{q})$$

We guess the anti-derivative $W_t(\mathbf{q}) = \frac{1}{2\hbar}\Omega(\pi_+ e^{-tA}\mathbf{q}, e^{-tA}\mathbf{q})$.

$$\frac{d}{dt}\Omega(\pi_+ e^{-tA}\mathbf{q}, e^{-tA}\mathbf{q}) = \Omega([A, \pi_+]e^{-tA}\mathbf{q}, e^{-tA}\mathbf{q})$$

Note that $\pi_+ + \pi_-$ is identity, so

$$A\pi_+ = (\pi_+ + \pi_-)A\pi_+ = \pi_+ A + \pi_- A\pi_+$$

thus $[A, \pi_+] = \pi_- A\pi_+$. π_+ and π_- are mutually adjoint with respect to Ω so our guess is correct.

$$W_t(\pi_+ e^{tA}\mathbf{q}) = \frac{1}{2\hbar}\Omega(\pi_+ e^{-tA}\pi_+ e^{tA}\mathbf{q}, e^{-tA}\pi_+ e^{tA}\mathbf{q}) = \frac{1}{2\hbar}\Omega(\mathbf{q}, e^{-tA}\pi_+ e^{tA}\mathbf{q})$$

We set $t = 1$ and get the final quantization formula

$$\hat{S}G(\mathbf{q}) = e^{\Omega(\mathbf{q}, S^{-1}\pi_+ S\mathbf{q})}G(\pi_+ S\mathbf{q})$$

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